2. K. Stewartson and P. G. Williams, "Self-induced separation," Proc. R. Soc. London, Ser. A, 312, No. 1509 (1969).
3. P. L. Krapivskii and V. Ya. Neiland, "Boundary-layer separation from the stationary surface of a body in a supersonic gas flow," Uch. Zap. Tsentr. Aérogidrodin. Inst., 13, No. 3 (1982).
4. V. I. Zhuk and O. S. Ryzhov, "Locally inviscid disturbances in a boundary layer with self-induced pressure," Dokl. Akad. Nauk SSSR, 263, No. 1 (1982).
5. Vik. V. Sychev, "Asymptotic theory of transient separation," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1979).
6. V. I. Zhuk, "Locally recirculating zones in a supersonic boundary layer on moving surface," Zh. Vychisl. Mat. Mat. Fiz., 22, No. 5 (1982).
7. O. S. Ryzhov and V. I. Zhuk, "Internal waves in the boundary layer with the self-induced pressure," J. Mec., 19, No. 3 (1980).

REFINEMENT OF THE EQUAL-AREAS LAW FOR UNSTEADY PLANE SHOCK WAVES OF MODERATE INTENSITY

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1. The equal-areas law, or Whitham's law, is often used in the theory of weak shock waves; it states that the integral of the flow velocity (excess density or pressure) is time-invariant (see [1-3]):

$$\int_L v\,dX = \text{const}, \tag{1.1}$$

where L is the wavelength, and $X = x - c_0 t$ is the accompanying coordinate. Equation (1.1) can be regarded as a conservation law for the simple-wave equation

$$v_t + \varepsilon v v_X = 0,$$

which is valid in the presence of discontinuities [1]. Here (and from now on) $\varepsilon = (1/2)(\gamma + 1)$. The equal-areas law describe the onset and development of an isentropic discontinuity in a simple wave and the laws governing the decay of plane shock waves.

Equation (1.1) follows from an analysis of the corresponding viscous problem. The simple-wave equation must be used with allowance for real dissipation (the Burgers equation in nonlinear acoustics) [2]:

$$v_t + \varepsilon v v_X - (b/2\rho_0) v_{XX} = 0 \tag{1.2}$$

[$b = \zeta + (4/3)\eta + (\gamma - 1)\kappa/c_p$ is the dissipation factor, η and ζ are the shear and bulk viscosity coefficients, and κ is the thermal conductivity]. An existing theorem [4] states that Eq. (1.2) has a unique conservation law, which coincides with Eq. (1.1).

We note that the Burgers equation is not a formal device in this case, but is derived from the complete system of equations of motion upon satisfaction of the natural asymptotic conditions [2, 5]

$$\partial/\partial X = O(1),\ \partial/\partial t = O(\mu);\ v,\ \rho' = O(\mu);\ \eta,\ \zeta,\ \varkappa = O(\mu), \tag{1.3}$$

where μ is a small parameter (the wave amplitude), and ρ' is the excess density. Conditions (1.3) have the physical significance that the traveling waveform varies slowly as a result of weak nonlinearity and dissipation.

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The rule (1.1) is valid in small principal-order terms with respect to the wave amplitude and has error $O(\mu^2)$. The solution of the Burgers equation also has uniform error $O(\mu^2)$ [5], even though the equation itself includes second-order small terms.

2. We now examine the possibility of refining Eq. (1.1) in second-order small terms. The correct formulation of the problem requires that the solution of the viscous problem be known with the same degree of accuracy. The solution of the Burgers equation is unsuitable for this purpose. We must therefore invoke the more precise evolution equation from the theory of finite-amplitude waves in a viscous heat-conducting polytropic gas [6]:

$$v_t + \varepsilon v v_X - \frac{b}{2\rho_0} v_{XX} = \frac{3}{4}\varepsilon \frac{b}{\rho_0 c_0} v_X^2. \tag{2.1}$$

Equation (2.1) is derived in exactly the same way as the Burgers equation, except that third-order small terms are included in the equations of motion, and its error is $O(\mu^4)$. The solution v itself has uniform error $O(\mu^3)$. The important thing is that all cubic nonlinear terms cancel out. The term on the right-hand side has the same form as the density of an irreversible entropy source in nonequilibrium thermodynamics [7] and characterizes a nonisentropic process.

Equation (2.1) is valid at large Reynolds number when divergent dissipative third-order terms of the form $b(v^2)_{XX}$, $b^2 v_{XXX}$, etc. are disregarded. These terms are vanishingly small in the limit $b \to 0$ and do not contribute to the shock condition or the conservation law. The term on the right-hand side, on the other hand, provides a finite third-order contribution to the condition at the shock discontinuity in the limit $b \to 0$ [1, 6].

Equation (2.1) has a conservation law of somewhat unconventional form [6]:

$$\int\limits_L \left(\exp\left(\frac{3}{2}\frac{\varepsilon v}{c_0}\right) - 1\right) dX = \text{const}, \tag{2.2}$$

but only the linear and quadratic terms of the series have any physical significance. Equation (2.2) is verified by direct substitution. Equation (2.1) must first be multiplied by $\exp[(3/2)\varepsilon v/c_0]$. Expansion in a series and the elimination of higher order terms give the required result:

$$\int\limits_L v\,dX + \frac{3}{4}\frac{\varepsilon}{c_0}\int\limits_L v^2\,dX = \text{const}. \tag{2.3}$$

Equation (2.3) is independent of the dissipation factor b and refines the equal-areas rule (1.1) in higher-order small terms. If the wave amplitude is small, the nonlinear term can be excluded. However, it yields a finite contribution for waves of moderate intensity.

In this situation the nonisentropicity of the flow is manifested in second-order small terms with respect to the wave amplitude. If the term on the right-hand side of Eq. (2.1) is eliminated, the nonlinear term in Eq. (2.3) is equal to zero. This dependence is similar to the expression for the velocity of the shock front, where a nonadiabatic correction is also observed in second-order rather than in third-order small terms [8].

The increase in order is attributable to the cumulative nature of the processes described by Eq. (2.1). Despite the fact that the nonisentropic correction is third-order small, it builds up over large distances (times) and ultimately causes the solution v to change in second-order small terms. Equation (2.1) specifically reflects this fact. Here we have an analogy with the nonlinear term in the Burgers equation or in the simple-wave equation. The nonlinear second-order increment also builds up with time and induces a variation of the solution in principal-order small terms.

3. Here we show that the rule (2.3) is indeed more precise than (1.1) for unsteady shock waves of moderate intensity. In the smooth domain the solution in either case has the form of a simple wave and is formulated by the method of characteristics.

The choice of initial disturbance is an unsteady shock wave in the form of a velocity discontinuity generated in air ($\gamma = 1.4$) at a certain distance from the center of a planar explosion [9]. Figure 1 shows the results of numerical calculations [9] at three different times (solid curves). Dimensionless variables are used here and elsewhere [9]; the quantity $r - \sqrt{\gamma}\tau$ (τ is dimensionless time) corresponds to a coordinate system moving with velocity

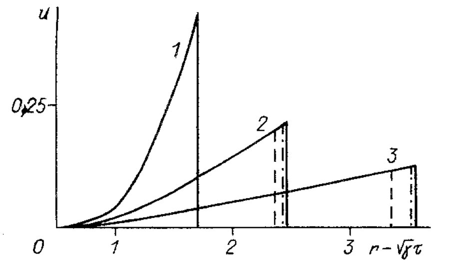

Fig. 1

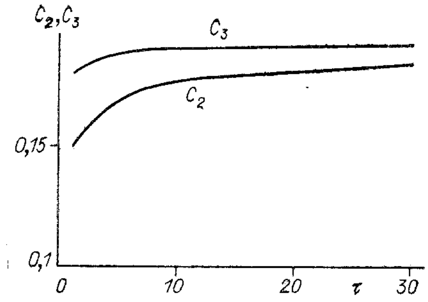

Fig. 2

c_0, and u is the dimensionless flow velocity. The three times correspond to pressure increments at the shock front $\Delta p = 0.66$ atm, 0.29 atm, and 0.16 atm, respectively.

We take the first profile as an initial condition and formulate an approximate solution in the smooth domain by the method of characteristics. The rule (1.1) or (2.3) must be used in order to determine the coordinate of the front. In dimensionless variables

$$\int_L u\,dy = C_2; \tag{3.1}$$

$$\int_L u\,dy + \frac{3}{4}\frac{\varepsilon}{\sqrt{\gamma}}\int_L u^2\,dy = C_3 \tag{3.2}$$

($y = r - \sqrt{\gamma}\tau$, and C_2 and C_3 are constants). In the determination of the coordinate of the front the definite integrals of u and u^2 are computed by the trapezoidal rule with a step not greater than 0.01L' (L' is the length of the interval of integration). The missing values of the integrands are found by graphical interpolation. The results of calculations according to Eqs. (3.1) and (3.2) are shown in Fig. 1. The dashed lines represent the shock front according to Eq. (3.1), and the dot—dash lines correspond to Eq. (3.2). We see that the rule (2.3) or (3.2) refines the position of the shock front and the amplitude of the velocity discontinuity. Table 1 gives the values of the relative error Δf of determination of the front coordinate (relative to the characteristic dynamic length r_0 [9]) and the error Δv of determination of the amplitude of the velocity discontinuity in comparison with the numerical solution. The index 2 indicates the result corresponding to the equal-areas rule, and the index 3 corresponds to the more precise equation (3.2); $\tau = 0.989$, 4.997, 15.93 correspond to the profiles represented by curves 1-3 in Fig. 1. It follows from these data that the correction to the equal-areas rule is very significant in the indicated range of intensities and diminishes the error at least by one fourth on the average.

It is instructive to analyze the behavior of the "constants" C_2 and C_3 for the numerical solution [9] at various times (see Table 1, which also gives the error with respect to the initial data). The approximate behavior of the $C_2(\tau)$ and $C_3(\tau)$ curves is illustrated in Fig. 2. Graphical interpolation is used because of a certain scatter of the tabulated values. The difference between C_2 and C_3 is particularly large for small τ, when the wave amplitude is still not too small. As time passes, the difference becomes less appreciable, and the C_2 and C_3 curves merge into one in the limit $\tau \to \infty$.

On the smooth parts of the profile the simple-wave approximation coincides with the numerical solution almost everywhere. The exception is a small zone in the tail of the wave at $\tau \gtrsim 15.93$, where the result is somewhat too low. This conclusion is consistent with [10], in which it is shown that the simple-wave approximation in the smooth domain can be used up to Mach numbers of the order of 1.5.

4. We now analyze Eq. (2.3) from the gasdynamic point of view. In gas dynamics (see, e.g., [1, 3]) the equal-areas rule follows from a solution of the simple-wave type with characteristic velocity

$$dx/dt = c_0 + \varepsilon v, \tag{4.1}$$

augmented by the expression for the weak shock velocity

$$dx_s/dt = c_0 + (1/2)\,\varepsilon\,(v_1 + v_2). \tag{4.2}$$

TABLE 1

	τ					
	0,989	1,993	4,997	7,967	15,93	29,90
$z = \frac{p_2 - p_0}{p_0}$	0,656	0,459	0,289	0,229	0,163	0,119
Δf_2, %	—	3,1	9,0	14	21	34
Δf_3, %	—	0,8	2,4	4,0	4,0	8,0
Δv_2, %	—	2,9	6,2	7,6	9,1	10,4
Δv_3, %	—	0,7	1,8	2,0	2,2	2,5
C_2	0,149	0,159	0,168	0,175	0,179	0,185
C_3	0,180	0,183	0,186	0,190	0,189	0,192
ΔC_2, %	—	6,7	13,4	17,5	20,1	23,7
ΔC_3, %	—	1,7	3,3	5,6	5,0	6,7

Equation (2.1) gives a more precise result in the limit b → 0 [6]:

$$dx_s/dt = c_0 + (^1/_2)\,\varepsilon\,(v_1 + v_2) + (^1/_8)\,\varepsilon^2\,[v]^2/c_0 \tag{4.3}$$

(c_0 is the equilibrium sound velocity in the rest medium). It is therefore reasonable to assume that the rule (2.3) can be substantiated by means of the simple-wave solution and the equation (4.3) for the velocity of a moderate-intensity shock wave.

It follows from the geometrical pattern of distortion of the flow velocity profile (see [3], p. 537) that the conservation law (2.3) can be rewritten

$$\int\limits_{v_1}^{v_2} (x - x_s)\,dv + \frac{3}{4}\frac{\varepsilon}{c_0}\int\limits_{v_1^2}^{v_2^2} (x - x_s)\,dv^2 = \text{const.} \tag{4.4}$$

The first integral is the difference in the areas intercepted by the shock wave, and the second integral is the same difference on the graph of the function v^2.

We differentiate Eq. (4.4) with respect to time, making use of Eqs. (4.1) and (4.3). Rejecting fourth-order small terms, we obtain

$$\varepsilon\int\limits_{v_1}^{v_2}\left(v - \frac{1}{2}(v_1 + v_2)\right)dv - \frac{1}{8}\frac{\varepsilon^2}{c_0}(v_2 - v_1)^2\int\limits_{v_1}^{v_2} dv + \frac{3}{4}\frac{\varepsilon^2}{c_0}\int\limits_{v_1^2}^{v_2^2}\left(v - \frac{1}{2}(v_1 + v_2)\right)dv^2 = 0. \tag{4.5}$$

It must be borne in mind that the difference $x - x_s$ vanishes everywhere at the limits of integration, even though they are variable. It is therefore sufficient to differentiate only the integrand [3]. The first integral (4.5) vanishes by virtue of the equal-areas rule. It is readily verified that the second and third integrals cancel one another, so that Eq. (4.5) holds identically.

Thus, the rule (2.3) follows from the adiabatic solution for a simple-wave and the expression (4.3) for the velocity of the shock front. The latter exhibits nonisentropic behavior; if we use only two mechanical conditions at the discontinuity for adiabatic flow, the third term in Eq. (4.3) will have a different form. This property is specific to the model (see Sec. 2) and implies that we are investigating only the growing nonisentropic correction reflected in second-order small terms. Cumulative growth does not occur on the smooth parts of the profile, and the simple-wave approximation remains unchanged. Subtler nonisentropic effects such as the reflected shock and entropy wake are third-order small and require special analysis [2, 6, 11].

5. The combined application of the simple-wave approximation with Eq. (4.3) has been discussed at length in the literature and comprises the substance of the well-known method of Friedrichs (see [8]). The rule (2.3) is probably more practical, especially in regard to complex disturbances. In particular, it leads to a new law of evolution of a sawtooth wave with balanced phases. This law entails asymmetrical distortion of the wave profile and has been observed [12] through a numerical analysis of the solutions of Eq. (2.1) in a low-viscosity medium.

Let us consider a sawtooth with the phases in direct sequence in an accompanying coordinate system. The asymptotic solution in the smooth domain in the limit $t \to \infty$ is $v = X/\varepsilon t$. Substituting this expression in Eq. (2.3), we find the coordinate of the discontinuity in the positive and negative phases:

$$s^{\pm} = \pm\sqrt{2\varepsilon C^{\pm} t} - \frac{1}{2}\frac{\varepsilon}{c_0}C^{\pm}.$$

The constants have the form

$$C^{\pm} = V \pm \frac{3}{4}\frac{\varepsilon}{c_0}W,$$

where V is the area of the positive or negative phase of the wave at the initial time, and W is the area under the graph of the function v^2 in the positive or negative phase.

Since $C^+ > C^-$, the length of the positive phase of the wave increases more rapidly than the length of the negative phase, so that $s^+ > |s^-|$. Similarly, the amplitude of the discontinuity in the positive phase of the wave exceeds the amplitude of the discontinuity in the negative phase. For a sawtooth with the phases in reverse sequence, the compression phase follows the rarefaction phase, and the discontinuity is located at the central point $X = 0$. The asymptotic solution is written [1]

$$v = (X - a)/\varepsilon t,\ s_0 < X < a;\ v = (X + a)/\varepsilon t,\ -a < X < s_0 \tag{5.1}$$

(2a is the wavelength; according to the equal-areas rule, the coordinate s_0 of the discontinuity is equal to zero at all times). The substitution of Eqs. (5.1) in (2.3) gives the more accurate value

$$s = \frac{3}{8}\frac{\varepsilon^2}{c_0}\frac{W}{a}t - \frac{1}{4}\frac{a^2}{c_0 t},$$

which shows that a sawtooth with the phases in reverse sequence is also distorted asymmetrically. The discontinuity shifts into the negative phase region and moves along the wave profile. If t is very large, the negative phase vanishes completely, and the wave itself is transformed into a monopolar pulse.

In application to the problem of a signal that is harmonic at the input, the rule (2.3) leads to the inception and growth of a phase shift and endows the flow velocity with a second-order small constant component in the sawtooth stage [6].

LITERATURE CITED

1. G. B. Whitham, Linear and Nonlinear Waves, Wiley-Interscience, New York (1974).
2. O. V. Rudenko and S. I. Soluyan, Theoretical Foundations of Nonlinear Acoustics, Consultants Bureau, New York (1977).
3. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, 2nd ed., rev., Pergamon Press, Oxford—New York (1987).
4. A. M. Vinogradov and I. S. Krasil'shchik, "A method for computing higher symmetries of nonlinear evolution equations and nonlocal symmetries," Dokl. Akad. Nauk SSSR, 253, No. 6 (1980).
5. S. Leibovich and R. Seebass, "Examples of dissipative and dispersive systems described by the Burgers and Korteweg—de Vries equations," in: Nonlinear Waves, Cornell Univ. Press, Ithaca (1974).
6. S. N. Makarov and B. S. Khamzina, "Evolution of a harmonic signal in the gasdynamic approximation of the theory of finite-amplitude plane waves," Akust. Zh., 34, No. 1 (1988).
7. I. Gyarmati, Non-Equilibrium Thermodynamics, Springer-Verlag, Berlin—New York (1970).
8. R. Courant and K. Friedrichs, Supersonic Flow and Shock Waves, Wiley, New York (1948).
9. Kh. S. Kestenboim, G. S. Roslyakov, and L. A. Chudov, Point Detonation: Computational Methods and Tables [in Russian], Nauka, Moscow (1974).
10. Y. Nakamura and Y. Nakashima, "Analysis of unsteady shock propagation generated by interaction with simple expansion waves," Tech. Rep. Kunamoto Univ., 36, No. 1 (1987).
11. S. N. Makarov and B. V. Filippov, "Calculation of finite-amplitude waves in the gasdynamic approximation," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2 (1986).
12. S. N. Makarov and B. S. Khamzina, "Numerical calculation of the evolution of strong pulses in the gasdynamic approximation," Vestn. Leningr. Univ., Ser. 1, No. 3 (1987).